Exercise 1 :

(i) $\bar{x}$ solution of (1) $\Rightarrow \begin{cases} \bar{x} \in C \\ f(\bar{x}) \leq f(x) \ \forall x \in C \end{cases}$

Suppose $(\bar{x}, r)$ w/ $r < f(\bar{x})$, solution of (2)

This would mean that $f(x) \leq r < f(\bar{x})$ which contradict $\bar{x}$ being a solution of (1)

Hence (x, r) is a solution of (2)

(ii) If $(\bar{x}, \bar{r})$ is a solution of (2) $\Rightarrow \begin{cases} \bar{r} \leq r \ \forall r \geq f(x) \text{ w/ } x \in C \\ f(\bar{x}) \leq \bar{r} \\ \bar{x} \in C \end{cases}$

Thus $\bar{r} \leq f(x) \ \forall x \in C$;

In particular, $\bar{r} \leq f(\bar{x})$, along with $f(\bar{x}) \leq \bar{r}$;

This means that $\bar{r} = f(\bar{x})$

Thus $\bar{x}$ is such that $\begin{cases} \bar{x} \in C \\ f(\bar{x}) = \bar{r} \leq f(x) \ \forall x \in C \end{cases}$

which is exactly optimality of problem (1).

Exercise 2 :

$$\min_x \|Ax - b\|_\infty$$

$$\overset{(Ex. 1)}{\iff} \begin{cases} \min\limits_{x, r} \ r \\ \text{s.t. } \|Ax - b\|_\infty \leq r \end{cases}$$

$\max_i |A_i x - b_i| \leq r$ (A_i: i-th row of A; b_i: i-th element of b)

$\Updownarrow$

$|A_i x - b_i| \leq r \quad \forall i$

$\Updownarrow$

$-r \leq A_i x - b_i \leq r \quad \forall i$

$\Updownarrow$

$A_i x - b_i \leq r \quad \forall i$

$A_i x - b_i \geq -r \quad \forall i$

$\Updownarrow$

$\begin{bmatrix} x \\ \vdots \\ r \end{bmatrix}$ ← concatenat° of x and r

$$\begin{bmatrix} A & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} \end{bmatrix} \leq b$$

$$\begin{bmatrix} A & \begin{matrix} +1 \\ \vdots \\ +1 \end{matrix} \end{bmatrix} \geq b$$

Define $v = (x, r) \in \mathbb{R}^{n+1}$

$$\iff \begin{cases} \min\limits_{v = (x, r)} \underbrace{\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}}_{c} \begin{bmatrix} x \\ r \end{bmatrix} \\ \text{s.t. } \underbrace{\begin{bmatrix} A & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} \\ -A & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} \end{bmatrix}}_{G} \begin{bmatrix} x \\ r \end{bmatrix} \leq \underbrace{\begin{bmatrix} b \\ \vdots \\ -b \end{bmatrix}}_{h} \end{cases}$$

Exercise 3:

$$\min_{x\in\mathbb{R}^n} \|Ax-b\|_1 = \sum_{i=1}^{m} |A_i x - b_i|$$

Direct Extension of Ex. 1

$(x,r) \in \mathbb{R}^n \times \mathbb{R}^m$

$$\iff \begin{cases} \min_{(x,r)} \sum_{i=1}^{m} r_i \\ \text{s.t. } |A_i x - b_i| \leq r_i \quad \forall i \end{cases}$$

$$\iff \begin{cases} \min_{(x,r)} \sum_{i=1}^{m} r_i \\ \text{s.t. } Ax - b \leq r \\ \quad\;\; Ax - b \geq -r \end{cases}$$

$$\iff \begin{cases} \min_{(x,r)} \underbrace{[0 \cdots 0\ 1 \cdots 1]}_{c} \begin{bmatrix} x \\ r \end{bmatrix} \\ \text{s.t. } \underbrace{\begin{bmatrix} A & -I_m \\ -A & -I_m \end{bmatrix}}_{G} \begin{bmatrix} x \\ r \end{bmatrix} \leq \underbrace{\begin{bmatrix} b \\ -b \end{bmatrix}}_{h} \end{cases}$$

Exercise 4:

$$p(y|X\theta) \propto e^{-\frac{\|y - X\theta\|_2^2}{2\sigma}}$$

so maximizing $\rightarrow$ amounts to minimizing $\|X\theta - y\|_2^2$

$$\begin{cases} \max_{\theta} \|X\theta - y\|_2^2 = (X\theta - y)^T (X\theta - y) = \theta^T X^T X \theta - y^T X \theta - \theta^T X^T y + \cancel{y^T y} \\ \text{s.t. } \|X\theta - y\|_\infty \leq \varepsilon \end{cases}$$

$-y^T X\theta - \theta^T X^T y \iff -(X^T y)^T \theta$

$$\iff \begin{cases} \max_{\theta} \frac{1}{2} \theta^T \underbrace{(2X^TX)}_{P} \theta + \underbrace{(-2X^Ty)}_{q}^T \theta \\ \text{s.t. } \underbrace{\begin{bmatrix} X \\ -X \end{bmatrix}}_{G} [\theta] \leq \underbrace{\begin{bmatrix} y + \varepsilon \\ -y + \varepsilon \end{bmatrix}}_{h} \end{cases}$$