Analysis of max-consensus algorithms in wireless channels

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Abstract

In this paper, we address the problem of estimating the maximal value over a sensor network using wireless links between them. We introduce two heuristic algorithms and analyze their theoretical performance. More precisely, i) we prove that their convergence time is finite with probability one, ii) we derive an upper-bound on their mean convergence time, and iii) we exhibit a bound on their convergence time dispersion.

I. INTRODUCTION

Wireless Sensor Networks are systems composed of scattered agents with limited power and computational abilities. These agents may acquire some data and communicate through a wireless link to some other agents. Their goal is to auto-organize in order to compute a function of the collected data in a distributed fashion [1]. For instance, if temperature sensors are deployed in a hostile environment (e.g. mountains) and one wants to know the average temperature in the region by looking at any sensor, a simple idea would be to make the sensors randomly wake up and average their value with another sensor so that they all converge to the average value of the initial measurements of the network. Namely, the sensors want to achieve *consensus* over the average value of the initial measurements. This problem was extensively studied in the past few years [2], [3], [4]. However, the average is not always the most useful value to share. Indeed, in some applications, the maximal value may be of greater interest.

For example, if a sensor network has to transmit information periodically (e.g. the average temperature of the region in the previous scheme) through a costly link, it would be of interest to elect the sensor which has the most power resource to operate that communication. To do so, one has to estimate the maximal amount of energy left in the sensor battery (along with their ID) in a distributed fashion using the wireless links between some of them. Another useful application deals with distributed access control by having the network elect a node to transmit. For instance, the agents that want to send information i) draw a number in a common window and then ii) reach consensus over the maximal value (and the ID of the associated agent). The sensor with the maximal value then sends its packet to the access point.

A simple way to estimate the maximum value would be to mimic the algorithm introduced for averaging [3]. The agents would wake up randomly and exchange their value with another reachable sensor randomly chosen; both sensors would then keep the maximum between their former and received values. However, since the communications between the sensors are wireless, it seems more natural for the initiating sensor to broadcast its value, and then the sensors which have received the information would update their value accordingly. In this work, we will analyze algorithms i) based on pairwise communications and ii) based on broadcast communications. Note that an averaging algorithm based on broadcast communications has been proposed in [5], but it does not perform well due to the non-conservation of the initial sum. This is not an issue for estimating the maximum value since the maximum value is preserved.

A distributed algorithm is thus relevant to estimating a maximum value over a network through wireless communications, which we propose to address hereafter. We prove the convergence of both above-mentioned algorithms and analyze their convergence speed.

This paper is organized as follows: models and algorithms are reported and linked with related works in Section II. In Section III, we derive our mathematical results. In Section IV, numerical illustrations are given. Concluding remarks are drawn in Section V.

II. MODELS AND ALGORITHMS

A. Assumptions on the wireless network

Consider a network of N sensors modeled as an undirected graph $G = (V, E)$ where V is the set of agents – the vertices of the graph – and E is the set of links between agents – the edges of the graph. We assume that each link is error-free. To indicate that a couple of agents (v, w) are *neighbors* (we also use the term *adjacent*), we use the notation $v \sim w$. For any set S, we denote its cardinality by $|S|$. Obviously, we have $|V| = N$. The set of neighbors of agent v is denoted $\mathcal{N}_v$. We also define $\partial(S)$, the set of edges going out of set S (that is, edges with one end in S and one end out of S) and

$\alpha_G = \min_S |\partial(S)|/|S|$ the vertex expansion. Each agent v has an initial scalar measurement $x_0(v)$. The set of all initial measurements is thus stacked in a unique vector $\mathbf{x}_0 \in \mathbb{R}^N$. The network is invariant over time and *connected*.

The network is assumed *asynchronous*, meaning that no common clock is available for the agents. Instead, each agent has its own clock and can initiate a communication with its neighborhood at clock ticks. Assuming communication time is small compared to the time between clock ticks, it makes sense (as usually done for other consensus-like algorithms [3], [5]) to assume the absence of collisions between communicating nodes. We also consider that the agent clocks are modeled by independent Poisson processes with intensity λ_v for agent v. It is then equivalent to have a global clock according to a Poisson process with intensity $\lambda = \sum_v \lambda_v$, and that each clock tick is then attributed to a given agent. The probability that agent v wakes up is equal to $p_v = \lambda_v/\lambda$. We will assume, for the sake of simplicity, that all intensities λ_v are the same, hence $\lambda_v/\lambda = 1/N$. We denote by $x_n(v)$ the value at agent v after n global clock ticks, while $\mathbf{x}_n$ denotes the vector of all values after n global clock ticks.

The goal for the network is to estimate the value $M(\mathbf{x}_0) \triangleq \max_{v \in V} x_0(v)$, in a *distributed* manner, that is, only using communications between adjacent nodes.

B. Algorithms

We propose two algorithms for achieving the task of estimating $M(\mathbf{x}_0)$. Both algorithms are inspired by those already developed to obtain the average-consensus.

The first algorithm is based on the exchange between the current values of two adjacent nodes chosen randomly in the following way.

RANDOM-PAIRWISE-MAX:

1) After the n-th clock tick, a node v wakes up[1].
2) v chooses a neighbor w uniformly in $\mathcal{N}_v$.
3) $x_n(v) = x_n(w) = \max\big(x_{n-1}(v), x_{n-1}(w)\big)$.

This algorithm is suitable for wired networks whereas it is clearly not optimal for wireless networks. Indeed, it does not rely on the broadcasting abilities of the wireless channel, in which all the neighbors receive the current value of v. Therefore, we propose a second algorithm that benefits from the broadcast nature of the wireless channel.

RANDOM-BROADCAST-MAX:

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[1]For the sake of clarity, the time index is dropped for the chosen nodes in this section.

1) After the n-th clock tick, a node v wakes up.

2) v broadcasts its current value to all its neighbors.

3) $x_n(w) = \max\big(x_{n-1}(w), x_{n-1}(v)\big)$ for $w \in \mathcal{N}_v$.

A similar algorithm has already been proposed for calculating the average (in that case, the max operator has to be replaced with the average one). Unfortunately, in the context of average-consensus, such an algorithm does not keep the sum constant along time, which prevents it from converging to the true value. As for the max-consensus, such an algorithm keeps the maximum value and so does not give rise to an undesirable behavior. Therefore the RANDOM-BROADCAST-MAX will be our flagship algorithm.

C. Link with existing works

To the best of our knowledge, in the framework of distributed computation, only [6] has focused on max computation. Actually, [6] has developed a general framework to compute a family of functions (including the maximum value) of the nodes' measurements in a distributed fashion. Compared to our set-up, this work has been done under continuous-time and synchronous clocks assumptions. It can nevertheless be adapted to our context (discrete-time and asynchronous clocks), but it will perform poorly since each node goes to the maximum value in an incremental way even if one of its neighbors has the maximum value. Therefore, our proposed algorithms are much more adapted to max computation.

The protocols presented here are similar to those used for rumor spreading, but the problems and solutions differ in several respects. In particular, they differ in which nodes transmit and update values in each time slot. In our framework, a single node v initiates a single transmission: in RANDOM-BROADCAST-MAX, all neighbors of v perform an update, and in RANDOM-PAIRWISE-MAX, a single neighbor w exchanges values with v and they both perform an update. By contrast, in rumor spreading problems, only nodes knowing the rumor are able to transmit, and all such nodes transmit in each time slot. For rumor spreading, nodes must know whether they have the rumor, whereas in our model, nodes do not know if they have the max value, so the time spent by each communication has to be taken into account. Furthermore, only one randomly-chosen node speaks at each clock tick. As a consequence, our communication system is inherently collision-free, whereas in rumor spreading synchronous communications are considered and so collisions may occur.

Only a few papers ([7], [8], [9], [10], [11]) have taken into account the broadcasting nature of the medium in the rumor spreading problem. But in all these papers, the communications are synchronous and so the main issue is collision between transmissions. Moreover, only the informed nodes wake up.

Hence, the major difficulty in the technical analysis lies in the study of the collisions. As a consequence, their set-up is different from ours, and their results do not hold in our context. In [12], the broadcasting nature of the channel is also considered in the so-called FLOOD-MAX algorithm. But the context is much simpler than ours since all the nodes wake up simultaneously and the communication is collision-free. Results obtained for this algorithm are clearly unsuited to our analysis.

All other papers dealing with rumor spreading ([13], [14], [15], [16], [17], [18], [19], [20]) focus on pairwise communication and so does not take advantage of the broadcasting nature of the channel. Consequently, their works and results can not be applied for the RANDOM-BROADCAST-MAX. In contrast, the proposed RANDOM-PAIRWISE-MAX seems closer to them. Actually, in most of these papers, only the informed nodes wake up and propagate their information to a randomly chosen neighbor, which differs significantly from our algorithm. However one algorithm, the so-called PUSH-PULL, is closely related to our algorithm. Indeed, at each clock tick, every informed node propagates its information to one of its neighbors randomly chosen (*push* step) whereas every uninformed node asks one of its neighbor for the information (*pull* step) [21], [22]. The update step is then clearly equivalent to those of RANDOM-PAIRWISE-MAX. Nevertheless one fundamental difference exists and prevents us from reusing results of the PUSH-PULL. Indeed, each node is active at each clock tick in the PUSH-PULL set-up whereas, in our set-up, one randomly-chosen node is active per clock tick. Consequently one node is active every N clock ticks *on average*. This requires using somewhat different tools to analyze the convergence.

Our problem and the proposed algorithms are thus novel, hence deserve our theoretical convergence analysis given hereafter.

III. PERFORMANCE ANALYSIS

We define convergence time τ as the first time when all the nodes share the same value, *i.e.*,

$$\tau \triangleq \inf\{n \in \mathbb{N} : \forall v \in V, \ x_n(v) = M(\mathbf{x}_0)\}. \tag{1}$$

Given an undirected graph $G = (V, E)$ with N nodes, one can define its $N \times N$ adjacency matrix A_G with entries: $a_G(v, w) = 1$ if $v \sim w$ and 0 otherwise. It is a symmetric matrix. We also introduce the $N \times N$ diagonal matrix D_G where the i-th diagonal entry is the degree of the node v_i, *i.e.*, $|\mathcal{N}_{v_i}|$. We denote $d_{\max}$ the maximum degree. The symmetric matrix $L_G = D_G - A_G$ is called the *Laplacian* of the graph G. Its eigenvalues are non-negative and its kernel has dimension 1 whenever the graph is connected. We denote by $\lambda_1, \ldots, \lambda_N$ the eigenvalues of L_G sorted in increasing order. The *diameter* of graph G is given by $\Delta_G = \max\{\ell(v, w) : (v, w) \in V^2\}$ where $\ell(v, w) = \inf\{m \in \mathbb{N} : [A_G]^m(v, w) > 0\}$ corresponds the minimum number of edges needed to connect v to w.

A. Random-Broadcast-Max

Theorem 1 asserts that all the sensors will share the maximum value after a finite number of clock ticks.

Theorem 1. *For* RANDOM-BROADCAST-MAX*, we have* $\tau < \infty$ *with probability* 1.

The proof is reported in Appendix A. This result is not at all surprising, and we would like now to have more information about the behavior of τ and, especially, about its mathematical expectation $\mathbb{E}[\tau]$ versus some characteristics of the operating graph.

Theorem 2. *For* RANDOM-BROADCAST-MAX*, one has*

$$\mathbb{E}[\tau] \le \beta, \quad where \; \beta = N\Delta_G + N(\Delta_G - 1)\log\left(\frac{N-2}{\Delta_G - 1}\right).$$

The proof is reported in Appendix B. Note that for complete graphs ($\Delta_G = 1$), the upper bound of Theorem 2 is tight since the time needed for propagating the *max* is the time needed for the *max*-informed node to wake up and communicate its value to all other nodes using only one broadcast communication, hence N in expectation. Moreover, for the ring graph, we can prove that $\mathbb{E}[\tau] = (N^2 - N)/2$ while the bound is equal to $N^2(1 + \log(2))/2$. By neglecting the term proportional to N, we observe that the mean and its bound are both scaled in N^2.

Let us consider the previous works on max propagation by using the broadcasting nature of the medium ([7], [8], [9], [10], [11]). Even if the framework is strongly different (see Section II-C), it is of interest to compare the performance bounds. When all the informed nodes wake up simultaneously and thus collide with each other, the best convergence time behaves like $\Delta_G \log(N/\Delta_G)$ [9]. Surprisingly, this is almost the same shape as ours except for a factor N.

Having an upper-bound on the expected convergence time is very useful, but does not provide information about the outliers, *i.e.*, the events for which the convergence time is extremely long. Therefore, in Theorem 3, we provide an upper bound for τ that holds with high probability.

Theorem 3. *For* RANDOM-BROADCAST-MAX*, with probability* $1 - \varepsilon$,

$$\tau \le \beta + N\Delta_G \left(\log\left(\frac{\Delta_G}{\varepsilon}\right) - 1\right)$$

The proof is reported in Appendix C. Let us consider the "toy" example of the complete graph. The extra time cost is equal to $N\log(1/\varepsilon)$, *i.e.*, $N\log N$ if $\varepsilon = 1/N$. Surprisingly, [15] obtained similar results although the two frameworks are strongly different.

B. Random-Pairwise-Max

A similar work can be done for the RANDOM-PAIRWISE-MAX. Actually the convergence can be proven by following the same approach as that given in Appendix A. In contrast, the proofs about mean convergence time and concentration rely on quite different tools and thus are introduced hereafter.

Theorem 4. *For* RANDOM-PAIRWISE-MAX*, one has*

$$\mathbb{E}[\tau] \leq \alpha, \quad where \ \alpha = N d_{\max} . \frac{h_{N-1}}{\lambda_2},$$

with the n-th harmonic number $h_n = \sum_{k=1}^{n} 1/k$.

The proof is reported in Appendix D. In order to illustrate the upper-bound given in Theorem 4, let us focus on the case where G is a complete graph. For such a graph, $d_{\max}/\lambda_2$ is of order $\mathcal{O}(1)$, hence our bound is of order $\mathcal{O}(N \log N)$. In the standard rumor spreading context, the bound is of order $\mathcal{O}(\log N)$ [23]. Once again, we pay an extra factor of order N for not knowing which nodes are informed or not.

Theorem 5. *For* RANDOM-PAIRWISE-MAX*, with probability $1 - \varepsilon$,*

$$\tau \leq \alpha \left(1 + \log \left(\frac{N}{\varepsilon} \right) . \left(1 + \sqrt{1 + \frac{1}{\log \left(\frac{N}{\varepsilon} \right)}} \right) \right).$$

The proof is reported in Appendix E. Note that for small ε, the RHS of Theorem 5 can be replaced with $\overline{T}_{\mathrm{RPM}}(1 + 2\log(N/\varepsilon))$. By taking $\varepsilon = 1/N$ (which is usual in the literature), we obtain $\overline{T}_{\mathrm{RPM}}(1 + 4\log(N))$. In [24], it is proven that τ for the PUSH-PULL is $\mathcal{O}(\alpha_G^{-1} \log^2(N)\sqrt{\log(N)})$ with probability $(1 - 1/N)$ where α_G is the vertex expansion. In Theorems 4 and 5, $\overline{T}_{\mathrm{RPM}}$ can in fact be replaced with $\overline{T}'_{\mathrm{RPM}} = N d_{\max} h_{N-1} \alpha_G^{-1}/2$ by applying the definition of α_G in Eq. (3). Therefore, τ for the RANDOM-PAIRWISE-MAX is $\mathcal{O}(N d_{\max} \alpha_G^{-1} \log^2(N))$ with probability $(1 - 1/N)$. Apart from the factor N (essentially due to our communication protocol), the trends offer strong similarities.

IV. NUMERICAL ILLUSTRATIONS

The proposed upper-bound for the expected convergence time and the convergence time dispersion have been evaluated on Random Geometric Graphs (RGG) which are well suited for modelling Wireless Sensor Networks. They consist in uniformly choosing N points (representing the nodes/sensors) in the unit square and then drawing an edge between every pair of sensors closer than a pre-defined radius r. By choosing $r = \sqrt{8\log(N)/N}$, connectivity is ensured with a high probability [25], [4].

In Figure 1, we plot the (empirical) mean number of communications for reaching convergence and the associated upper-bounds (given by Theorems 2 and 4) for each proposed algorithm versus the number

of sensors N. We observe that the RANDOM-BROADCAST-MAX outperforms the RANDOM-PAIRWISE-MAX. When the network size increases, the upper-bounds become quite pessimistic due to the various used simplifications (in the case of RANDOM-BROADCAST-MAX, we rely on the spanning tree instead of the whole graph and we broadcast the information layer per layer; in the case of RANDOM-PAIRWISE-MAX, we use Cheeger's inequality and the approximation $1/d_{\max}$).

As the RANDOM-BROADCAST-MAX is much more interesting in terms of performance, we hereafter focus on it exclusively. In Figure 2, we plot the histograms of the convergence time as well as the upper-bounds for the convergence with probability $1 - 1/N$ (given in Theorem 3) when $N = 40$.

V. CONCLUSION

We have proposed two algorithms for estimating the maximum value in wireless sensor networks. The convergence times of these algorithms have been analyzed in depth. Furthermore, our problem is close to the rumor spreading problem, and we show that roughly speaking we pay a factor about the size of the network for not knowing which nodes have the information (maximum value or rumor).

APPENDIX A

PROOF OF THEOREM 1

Let $K_n = \{v \in V : x_n(v) = M(x_0)\}$ be the set of nodes sharing the maximum at time n. Let $X_n = |K_n|$ be the cardinal of K_n and $Y_n = \delta_{\{X_{n+1} > X_n\}}$ be the random variable equal to 1 when $X_{n+1} > X_n$ and 0 otherwise.

As K_n is a nondecreasing sequence of subsets of V, X_n is a nondecreasing sequence of integers bounded by N and so X_n converges to a random variable $X_\infty \le N$. Hence, we have that $\sum_{n=1}^{\infty} Y_n < X_\infty \le N$ and, taking expectations, $\sum_{n=1}^{\infty} \mathbb{E}[Y_n] < \infty$.

Whenever $X_n < N$, using the graph connectedness, there is at least one couple $(v, w) \in K_n \times (V - K_n)$ such that $v \sim w$. The probability that v informs w with $M(\mathbf{x}_0)$ is greater than $p = 1/(N d_{max}) > 0$ (with both algorithms and for every time n). Then, for all $n > 0$

$$\mathbb{E}[Y_n] \ge \mathbb{P}[Y_n = 1, X_n < N] = \mathbb{P}[Y_n = 1 | X_n < N]\mathbb{P}[X_n < N] \ge p\,\mathbb{P}[X_n < N].$$

Then, by summing over $n > 0$, we have

$$p \sum_{n=1}^{\infty} \mathbb{P}[X_n < N] \le \sum_{n=1}^{\infty} \mathbb{E}[Y_n] < \infty.$$

Thanks to the Borel-Cantelli lemma, we know that $\mathbb{P}[X_n < N,\ \text{infinitely often}\ ] = 0$. Hence, there is a finite time τ such that $X_\tau = N$ almost surely.

APPENDIX B

PROOF OF THEOREM 2

We assume for the sake of simplicity that one single node, say $v^{(0)}$, has the maximum at time $n = 0$. Let us partition set V according to nodes' distances from $v^{(0)}$:

$$L_i = \{v \in V : d(v^{(0)}, v) = i\}, \quad i \in \mathbb{N}$$

One has $V = \cup_{i=0}^{\Delta_G} L_i$ and $L_i \cap L_j = \emptyset$ for $i \neq j$. We define the (random-variable) times: $t_0 = 0$, and $t_i = \inf\{n \geq t_{i-1} : \forall v \in L_i, x_n(v) = M(\mathbf{x}_0)\}$. We denote by $\mathcal{F}_n$ the σ-algebra spanned by the nodes sharing the maximum values at time n. Using the same proof framework as in the standard coupon collector problem (see e.g. [26]), it is easy to show that $\mathbb{E}[t_{i+1} - t_i | \mathcal{F}_{t_i}] \leq N h_{|L_i|}$. The term $\mathbb{E}[t_{i+1} - t_i | \mathcal{F}_{t_i}]$ corresponds to the duration to completely fill up layer $(i + 1)$ given the nodes sharing the maximum value at time t_i, *i.e.*, given at least that layer i was already filled up. Therefore we have

$$\mathbb{E}[\tau] \leq \sum_{i=0}^{\Delta_G - 1} \mathbb{E}[t_{i+1} - t_i | \mathcal{F}_{t_i}] \leq \sum_{i=0}^{\Delta_G - 1} N h_{|L_i|} .$$

By using the inequality $h_n \leq \log(n) + 1$ and the fact $|L_0| = 1$, we obtain

$$\mathbb{E}[\tau] \leq N \left(\Delta_G + \sum_{i=1}^{\Delta_G - 1} \log |L_i| \right) .$$

Using $\sum_{i=1}^{n-1} \log x_i \leq (n-1) \log(\frac{1}{n-1} \sum_{i=1}^{n-1} x_i)$, with $x_i = |L_i|$ and $n = \Delta_G$ concludes the proof.

APPENDIX C

PROOF OF THEOREM 3

Let $A_i^v(t)$ be the event that the node v (belonging to layer L_i) is not switched on after t iterations. So $\mathbb{P}[A_i^v(t)] = \left(\frac{N-1}{N}\right)^t$. When the event $t_{k+1} - t_k \geq t$ occurs, we know that the event $\cup_{v \in L_{i+1}} A_i^v(t)$ also occurs. Therefore $\mathbb{P}[t_{i+1} - t_i \geq t] \leq \mathbb{P}(\cup_{v \in L_{i+1}} A_i^v(t))$. By using the Union bound and the fact that $0 \leq 1 - y \leq \exp(-y)$ for $y \in [0, 1]$, one can prove that the probability that, after t iterations, some of the nodes of L_i still have not talked is as follows

$$\mathbb{P}[\cup_{v \in L_{i+1}} A_i^v(t)] \leq \sum_{v \in L_i} \exp \left(-\frac{t}{N} \right) . \tag{2}$$

For any $\varepsilon > 0$, by choosing $t_\varepsilon = N \log |L_i| + N \log(\Delta_G / \varepsilon)$, we then get

$$\mathbb{P} \left[t_{i+1} - t_i \geq N \log |L_i| + N \log \left(\frac{\Delta_G}{\varepsilon} \right) \right] \leq \frac{\varepsilon}{\Delta_G}.$$

By using once again the Union bound, we find the final result.

APPENDIX D

PROOF OF THEOREM 4

The definition of K_n is given at the beginning of Appendix A. In the context of RANDOM-PAIRWISE-MAX, one has $|K_n| \leq |K_{n+1}| \leq |K_n| + 1$. Here, our objective is to exhibit a tight evaluation of the probability that the sequence $|K_n|$ is strictly increasing at time n.

$$\mathbb{P}[|K_{n+1}| = |K_n| + 1 \mid K_n] = \mathbb{P}[v_n \in K_n, w_n \notin K_n | K_n, v_n \sim w_n] = \mathbb{P}[\{v_n, w_n\} \in \partial K_n | K_n, v_n \sim w_n].$$

The selection algorithm of an edge is as follows: choose v_n uniformly over V, then w_n uniformly over $\mathcal{N}_{v_n}$ and independently of the past, or vice-versa. Therefore, for any edge e, we have $\mathbb{P}[\{v_n, w_n\} = e] \geq 2/Nd_{\max}$ which implies that

$$\mathbb{P}[\{v_n, w_n\} \in \partial K_n \mid K_n, v_n \sim w_n] \geq 2\frac{|\partial K_n|}{Nd_{\max}}.$$

For any subset S of V, the following inequality, called Cheeger's inequality, holds

$$\frac{|\partial S|}{|S|} \geq \lambda_2 . \left(1 - \frac{|S|}{N}\right) \tag{3}$$

where $\partial S \triangleq \big\{\{v, w\} \in E : v \in S, w \notin S\big\}$ is the boundary of S. More details are available in [27].

Using Cheeger's inequality, we obtain

$$\mathbb{P}[\{v_n, w_n\} \in \partial K_n \mid K_n, v_n \sim w_n] \geq \frac{2\lambda_2}{N^2 d_{\max}}(N - |K_n|)|K_n| \tag{4}$$

As in Appendix B, assuming, for the sake of simplicity, that initially one single node has the maximum value, consider the stopping times: $\tau_i = \inf\{n \in \mathbb{N} : |K_n| = i\}$, so that $\tau_1 = 0$ and $\tau = \sum_{i=1}^{N-1}(\tau_{i+1} - \tau_i)$ (if more than one node have the maximum value at time 1, one just has to start at $i > 1$). Let L_n be equal to the random variable $|K_{n+1}| - |K_n|$ given $|K_n|$. L_n is a Bernoulli distribution of parameter p_n. From Eq. (4), we have $p_n \geq (2\lambda_2/N^2 d_{\max}).(N - |K_n|)|K_n|$. As $(\tau_{i+1} - \tau_i)$ is the number of iterations needed to increment $|K_n|$ when $|K_n| = i$, or equivalently, the number of trials on L_n for obtaining the value 1 when $|K_n| = i$, the random variable $(\tau_{i+1} - \tau_i)$ is geometrical distributed with parameter $p_i \geq \pi_i$ with $\pi_i = (2\lambda_2/N^2 d_{\max}).(N - i)i$. As a consequence, $\mathbb{E}[\tau_{i+1} - \tau_i] \leq 1/\pi_i$. We thus have

$$\mathbb{E}[\tau] \leq \frac{N^2 d_{\max}}{2\lambda_2} \sum_{i=1}^{N-1} \frac{1}{(N - i)i} = \frac{Nd_{\max}}{\lambda_2} \sum_{i=1}^{N-1} \frac{1}{i} ,$$

which after some simple algebra leads to the result.

APPENDIX E

PROOF OF THEOREM 5

We use the same notations as in Appendix D. The random variable $\tau_{i+1} - \tau_i$ is geometric-distributed with parameter $p_i \geq \pi_i$. As a consequence, $\tau_{i+1} - \tau_i$ is stochastically dominated by a geometric distribution with parameter π_i denoted by Y_i, which means that the cdf of $\tau_{i+1} - \tau_i$ is smaller than the cdf of Y_i at any point [28]. By using Chernoff's bound for geometric random variable, we have, for any $\delta > 0$,

$$\mathbb{P}\left[\tau_{i+1} - \tau_i \geq \frac{1+\delta}{\pi_i}\right] \leq \mathbb{P}\left[Y_i \geq \frac{1+\delta}{\pi_i}\right] \leq \exp\left(-\frac{\delta^2}{2(1+\delta)}\right).$$

Let ε be any positive value. Selecting δ_ε as the smallest positive term such that $\exp\left(-\delta_\varepsilon^2/(2(1+\delta_\varepsilon))\right) \leq \varepsilon/N$ leads to $\delta_\varepsilon = \log(N/\varepsilon)(1 + \sqrt{1 + 1/\log(N/\varepsilon)})$. So, we have $\mathbb{P}[\tau_{i+1} - \tau_i \geq (1+\delta_\varepsilon)/\pi_i] \leq \varepsilon/N$. Then, by using the Union bound, we have $\mathbb{P}[\tau \geq (1+\delta_\varepsilon)(\sum_i 1/\pi_i)] \leq \varepsilon$ which concludes the proof.

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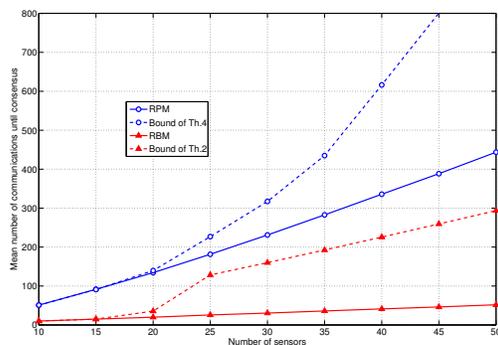
Fig. 1. (Empirical) mean number of communications for reaching convergence and the associated upper-bounds versus N.
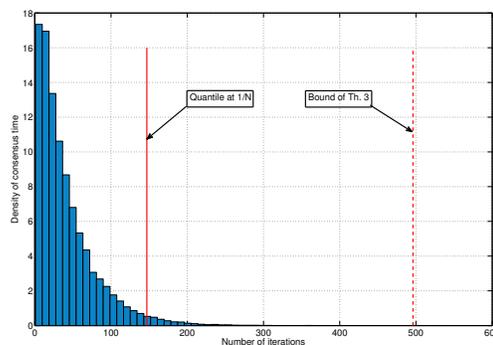
Fig. 2. Histogram of the convergence time and upper-bounds associated with probability $(1 - 1/N)$ for the RANDOM-BROADCAST-MAX when $N = 40$.