Sample Exercise - Nonsmooth Convex Optimization methods

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The notes in italics are simply hints for a possible answer and do not constitute a full response to the questions

Consider the $\mathbb{R} \to \mathbb{R}$ function

$$g : x \mapsto \begin{cases} \max(|x|, 2|x| - 1) & \text{if } |x| \leq 2 \\ +\infty & \text{else} \end{cases} \tag{1}$$

1. Is the function convex, proper, lower semi-continuous?
   it is easy to show that the epigraph of the function is a closed convex non-empty polytope. Another possiblity is to use the definitions directly.

2. Write the subdifferential at the minimum and show that this point verifies the first order optimality conditions.
   the minimum is 0, $\partial g(0) = [-1, 1]$, *the optimality condition* $0 \in \partial g(0)$ *is verified.*

3. Suppose that you only have access to $g(x)$ and $v \in \partial g(x)$ at a query point x. What algorithm would you use to minimize g? (several answers possible, justify yours)
   Since this is a nonsmooth convex function on a bounded set, a cutting plane method would be performing.

4. Compute the proximity operator of g for some stepsize $\gamma > 0$. (maybe the most difficult question)
   Compute the subdifferential of g at all points in $[-2, 2]$ *and write the optimality conditions of the prox (as for the l1 norm in the course). You will get that the prox is equal to* -2 *for inputs in* $(-\infty, -2 - 2\gamma]$*, then is linear up to reaching* -1 *which is the output for* $[-\gamma - 1, -2\gamma]$ *(a case has to be done on the value fo* γ*). This is a bit tedious but should be straightforward. Take* $\gamma < 1/2$ *to simplify things.*

5. What are the fixed points of the proximity operator? How would you minimize g using this proximity operator?

the only fixed point of the proximity operator is the minimum of the function, ie. 0.

6. Give an algorithm to minimize $a(x-1)^2 + g(x)$.
   Since we know how to compute the proximity operator of g, *and* $a(x-1)^2$ *is* $2a$*-smooth, one can use the proximal gradient.*