Univ. Grenoble Alpes
Master 1 AM
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Operations Research – Complementary

Final Exam

2021/2022
Duration: 2h

The exercises are independent. The questions all have the same weight in the grade.

EXERCISE 1: (8 Pts)

1. What are the properties of the graph represented by the following adjacency matrix? (number of vertices, edges, directed or not, connected or not)

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

2. The eigenvalues of A are:

$$\begin{array}{r} -1.4812 \\ -1.0000 \\ 0.3111 \\ 2.1701 \end{array}$$

   Is this matrix irreducible? primitive?

3. The entries of $(I + A)^2$ are all positive, what does this mean? How is this related to graph represented by A?

4. Show that it is not possible for a planar graph to have 6 vertices, 10 edges and 5 faces. Change one number (vertices, edges, or faces) to make it planar and draw the resulting planar graph.

EXERCISE 2: (8 Pts)

1. Consider a 2-player game with 3 actions for each player. How many pure Nash Equilibria can exist?

2. What is the pure Nash Equilibrium of the following game?

		Player 2		
		A	B	C
Player 1	a	$(3,1)$	$(2,3)$	$(10,2)$
	b	$(4,5)$	$(3,0)$	$(6,4)$
	c	$(2,2)$	$(5,4)$	$(12,3)$
	d	$(5,6)$	$(4,5)$	$(9,7)$

3. Depending on the value of the parameter $x \in \mathbb{R}$, give the pure and mixed Nash Equilibria for the following game:

		Player 2	
		A	B
Player 1	A	$(0.5, 0.5)$	$(0,1)$
	B	$(1,0)$	$(\frac{1-x}{2}, \frac{1-x}{2})$

First, for the pure NEs. (A,A) cannot be a pure NE. (B,A) and (A,B) are pure NEs if $x \geq 1$. Finally, (B,B) is a pure NE if $x \leq 1$.

Second, for mixed NEs. The indifference equation between the choices of player 2 with player 1 playing $yA + (1-y)B$ leads to

$$0.5y = y + 0.5(1-x)(1-y)$$

and thus $y = \frac{x-1}{x}$ which is well defined and in $[0,1]$ if $x \geq 1$. Thus, since the game is symmetric, $(\frac{x-1}{x}A + \frac{1}{x}B; \frac{x-1}{x}A + \frac{1}{x}B)$ is the mixed NE for $x \geq 1$. If $x = 1$, $y = 0$ and thus B is always chosen by player 1. Thus, for $x < 1$, the only NE is the pure one: (B,B).

4. In a game of penalty, both the Shooter and the Goalkeeper have to choose Left or Right. Then, if both choose Left, or both choose Right, the Goalkeeper stops the ball and gains +1 while the Shooter gains -1 (or losses 1). If they choose opposite directions, the Shooter gains +1 and the Goalkeeper gains -1. What are the properties of this game (number of players, actions, payoffs of the players)? How can you compute the mixed Nash equilibrium of this game? (Several responses are possible, you are not expected to give the result, just the method)
   There are 2 players, (Shooter&GK), the possible actions are Right and Left for both. This is a zero sum game (the payoff of the Shooter is the opposite of the one of the GK. the payoff of the shooter is $g(R,R) = g(L,L) = -1$ and $g(R,L) = g(L,R) = +1$.

   Since this is a zero sum, finite, two-player game, the NE and the min-max equilibrium are the same (Theorem 2.31). You can thus compute the sought equilibrium by getting the NE by indifference equations as in question 3, or by solving the max-min Linear Program, or using a algorithmic solution like Extragradient/Mirror-Prox.

EXERCISE 3: (4 Pts)

1. We want to transport a distribution $\alpha = 0.2 \times \delta_0 + 0.8 \times \delta_1$ (ie. a mass of 0.2 at $x = 0$ and a mass of 0.8 at $x = 1$) to a distribution $\beta = 0.3 \times \delta_2 + 0.7 \times \delta_5$. Does the Monge problem have a solution? Write a coupling corresponding to any (not necessary optimal) Kantorovich transport plan. What properties does it have to verify to be admissible?

2. Give the optimal transport plan to the problem of question 1 in the sense of Kantorovitch with cost $c(x,y) = |x-y|$.