CHAPTER 1 Spectral Graph Theory

This chapter is devoted to the study of eigenvalues/vectors of matrices associated with graphs.

Why?

- Combinatorial properties of graphs are linked to eigenelements
- Nice (and useful!) mathematical results

This part is mainly based on:

- The course of Dan Spielman at Yale: `http://www.cs.yale.edu/homes/spielman`
- The "Horn & Johnson" book (Horn and Johnson, 2012)

1.1 Graphs

A graph $\mathcal{G}$ is a collection of *nodes/vertices* V linked by *edges/links* E; so we note $\mathcal{G} = (V, E)$:

- V is usually $\{1, .., n\}$ where n is the number of vertices.
- E has the form $\{(i_k, j_k)\}_k$ where $(i_k \in V, j_k \in V)$ represents the (ordered) edge between i_k and j_k. The edges may also have *weights* and thus write $(i_k \in V, j_k \in V, w_k \in \mathbb{R}^+)$. In none are precised, these weights are implicitly assumed to be equal to 1.

The edges set represents the connections and is thus the most significative part of the graph. If all edges are bi-directionals (i.e. $(i, j) \in E \Leftrightarrow (j, i) \in E$), the graph is said *undirected*, as opposed to the general, *directed* case (the word digraph is sometimes used).

For a vertex $i \in V$, we define its *neighbors* as $\mathcal{N}_i := \{j \in V : (i, j) \in E\}$ and its *degree* $d_i = |\mathcal{N}_i|$. Also, we say that there is a *path* from i to j if there is a sequence of connected edges $\{(i, l_1), (l_1, l_2), (l_2, l_3), .., (l_k, j)\} \in E^k$.

Definition 1.1 (Connected Graph)**.** An undirected graph is said to be *connected* if for any pair (i, j) there is a path from i to j.
A directed graph is said to be *strongly connected* if for any pair (i, j) there is a path from i to j and path from j to i.

Examples of graphs:

- Social Networks: people are vertices, friendship/follow are links; typical problem: cluster communities.

- Power Grid: producers/consumers are vertices, cables are links (with their capacity as weight); typical problem: optimize the flow.
- Protein interaction, etc.

1.2 The Adjacency matrix

A graph can be equivalently represented by its *adjacency matrix*.

Definition 1.2. For a graph $\mathcal{G} = (V, E)$ with $n = |V|$ vertices, the adjacency matrix is the $n \times n$ matrix defined as

$$A_{ij} = \begin{cases} 1 & \text{if } (j,i) \in E \\ 0 & \text{otherwise} \end{cases} .$$

and in the case of a weighted graph

$$A_{ij} = \begin{cases} w & \text{if } (j,i,w) \in E \\ 0 & \text{otherwise} \end{cases} .$$

Example 1.3.

We can directly observe some properties:

- A is symmetric if and only if the graph is undirected.
- A is non-negative.
- The position of the *zeros* is important.

Goal: Investigate the spectral properties of Adjacency matrices (and related ones). Most of the content is valid for any non-negative matrix *but* the graph interpretation sheds some light on the developed notions.

1.2.1 *Power of Adjacency Matrices*

Example 1.4.

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 3 & 2 & 4 & 1 \\ 4 & 4 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{bmatrix} \qquad A^4 = \begin{bmatrix} 12 & 13 & 17 & 6 \\ 13 & 12 & 17 & 6 \\ 17 & 17 & 14 & 11 \\ 6 & 6 & 11 & 2 \end{bmatrix}$$

then all A^k $(k \geq 4)$ have positive coefficients.
The eigenvalues of A are $-1.45, -1, 0.31, 2.17$.

Proposition 1.5. *$\left(A^k\right)_{i,j}$ is the number of paths that go from i to j in exactly k steps.*

Proof. as an exercise. □

Example 1.6.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A^4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = A$$

The eigenvalues of A are $-0.5 + 0.86i, -0.5 - 0.86i, 1$, notice the pair of complex eigenvalues with magnitude 1.

If we add a small modification:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad A^4 = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

then all A^k $(k \geq 4)$ have positive coefficients.
The eigenvalues of A are $1.46, -0.23 + 0.8i, -0.23 + 0.8i$, notice the pair of complex eigenvalues with magnitude < 1.

Intuition: Power of adjacency matrices enable to investigate the connectedness of a graph.

1.2.2 *First Spectral Results: Non-negative Matrices*

[Cor. 8.1.19](Horn and Johnson, 2012)

Proposition 1.7. *Let $A, B \in \mathcal{M}_n$. If $0 \leq A \leq B$, then $\rho(A) \leq \rho(B)$.*

Proof. as an exercise. □

This means that by adding new links, we increase the spectral radius.

Lemma 1.8. *Let $A \in \mathcal{M}_n > 0$, then*

a) if the row sum is constant; $\rho(A) = |||A|||_\infty = \sum_j A_{ij}$;

b) if the column sum is constant; $\rho(A) = |||A|||_1 = \sum_i A_{ij}$;

c) $\min_i \sum_j A_{ij} \leq \rho(A) \leq \max_i \sum_j A_{ij}$;

d) $\min_j \sum_i A_{ij} \leq \rho(A) \leq \max_j \sum_i A_{ij}$.

Proof. TODO □

If the number of (outgoing/ingoing) edges, i.e. the degree, is constant across vertices, then it is equal to the spectral radius of A.

1.2.3 *Perron-Frobenius theorem for positive matrices*

When a matrix is positive[1], we have the well-known Perron-Frobenius theorem (1907 and 1912).

[1] positive elementwise, not positive definite

Theorem 1.9 (Perron-Frobenius). *Let $A \in \mathcal{M}_n > 0$, then*

a) *$\rho(A)$ is an eigenvalue (called Perron/leading/dominating eigenvalue) with multiplicity one, and all the other eigenvalues have a smaller modulus;*

b) *there is a* positive *eigenvector v (called Perron vector) associated with $\rho(A)$ and no positive eigenvectors for the other eigenvalues;*

c) $\lim_{k\to\infty} \frac{A^k}{\rho(A)^k} = vw^\top$ *where* $\begin{cases} Av = \rho(A)v \\ A^\top w = \rho(A)w \\ w^\top v = 1 \end{cases}$.

Proof. See wikipedia or Chap. 8.5 in (Horn and Johnson, 2012). □

Example 1.10 (Application: PageRank). A method to rank webpages designed in 1997 by Larry Page, co-founder of Google.

Graph of webpage connections:

- *not* strongly connected;
- A is not irreducible.

PageRank:

1. Normalize A so that the columns sum to 1;
2. $M = (1-\alpha)A + \alpha\frac{1}{n}\mathbf{1}\mathbf{1}^\top$ with α[2] $\in (0,1)$ is a positive matrix;
3. Eigenvalues of M: $1, -0.85, 0.425i, -0.425i, 0$. $\rho(M) = |||M|||_1 = 1$ since the columns sum to 1;
4. The Perron vector[3] (associated with 1) is $0.29, 0.12, 0.36, 0.19, 0.03$ hence the order $3, 1, 4, 2, 5$.

[2] $\alpha = 0.15$ in the original version

[3] Can be computed with the power method for instance.

1.2.4 *Primitivity and irreducibility*

Definition

Definition 1.11. A matrix $A \in \mathcal{M}_n \geq 0$ is said to be

- *irreducible* if $(I + A)^{n-1} > 0$;
- *primitive* if $A^m > 0$ for some m.

Proposition 1.12. *A matrix $A \in \mathcal{M}_n \geq 0$ is* primitive *if and only is $\rho(A)$ is the unique eigenvalue of maximal modulus.*

Example 1.13. See before

Theorem 1.14. *Let $A \in \mathcal{M}_n \geq 0$, the following are equivalent:*

a) *A is irreducible;*

b) *The graph $\mathcal{G}$ associated with A is strongly connected;*

c) *The mask of A is irreducible.*

Proof. Since $I + A$ is the adjacency matrix of $\mathcal{G}$ to which we added self-edges (i to i). For any i, j, $(I + A)^{n-1}_{ij} > 0$ means that there is a path of length n between i and j (since we allowed self edges, and a path of size n would contain two identical vertices, it just means a path); which is the definition of strong connectivity. The converse is immediate. Finally, the weights of the edges play no role hence the last point. □

Perron-Frobenius theorem for irreducible matrices

Observing Definition 1.11 and Theorem 1.14, it is natural to ask for a version of Perron-Frobenius for irreducible matrices.

Theorem 1.15 (Perron-Frobenius for irreducible matrices)**.** *Let $A \in \mathcal{M}_n \geq 0$ be irreducible, then*

a) $\rho(A)$ is an eigenvalue with multiplicity one;

b) there is a positive *eigenvector v associated with $\rho(A)$.*

Proof. See wikipedia or Chap. 8.5 in (Horn and Johnson, 2012). □

Most important change: $\rho(A)$ may *not* be the only eigenvalue of maximal modulus. But how many eigenvalues with maximal modulus can there be?

Definition 1.16. The period h of an irreducible matrix A is the greatest common divisor of $\{m : (A^m)_{ii} > 0\}$ i.e. the lengths of the directed paths in $\mathcal{G}$.

Proposition 1.17. *Let $A \in \mathcal{M}_n \geq 0$ be irreducible with period h. There are h eigenvalues of modulus $\rho(A)$:*

$$\{\exp^{2i\pi p/h} : p = 1, .., h\}.$$

Remark 1.18. If you add a self-loop (i.e. $A_{ii} > 0$ for some i), then $h = 1$, thus $\rho(A)$ is the unique eigenvalue of modulus $\rho(A)$. This means that A is primitive! ◁

Example 1.19. Cycle of 3: eigenvalues $-0.5 + 0.86i, -0.5 - 0.86i, 1$ all of modulus 1. Adding one self-loop we get $1.46, -0.23 + 0.8i, -0.23 - 0.8i$ (the complex eigenvalues have a modulus of 0.8).

Power method

As already seen in the PageRank example, the power method consists in iteratively applying a square matrix A:

$$x_{k+1} = \frac{Ax_k}{\|Ax_k\|}. \tag{Power method}$$

This method will converge to an eigenvector of the maximal eigenvalue provided that it is i) the unique eigenvalue of maximal modulus; and ii) positive and real. The second point is true for irreducible or primitive matrices. However, the first is only verified for primitive matrices. We thus have the following result.

Theorem 1.20. *Let $A \in \mathcal{M}_n \geq 0$ be a primitive matrix. Then, the Power method initialized with $x_0 \sim \mathcal{N}(0, I)$ converges to an eigenvector for the eigenvalue $\rho(A)$.*

Proof. See wikipedia for instance. The general idea is to use the eigendecomposition (or the Jordan form) and observe that apart from the top left coefficients, everything else vanishes. □

- The initialization is important: less than the distribution, the initial vector shall not be orthogonal to the sought eigenvector. Thus, might as well initialize randomly to be sure.
- If the matrix is only irreducible the power method will oscillate between the eigenspaces corresponding to the h eigenvalues of modulus $\rho(A)$ (see Proposition 1.17).

1.3 Applications

1.3.1 Markov Chains

A (discrete-time homogeneous) Markov chain is a sequence of random variables (X_k) where the distribution $x_{k+1} \in \mathbb{R}^n$ at time $k+1$ only depends on the distribution at time k and a transition matrix between the n states of the chain.

Mathematically, $x_k(i)$ is the probability that X_k is at state i ($X_k = i$) at time k. And $M \in \mathcal{M}_n \geq 0$ is defined as $M_{ij} = \mathbb{P}[X_{k+1} = i | X_k = j] \in [0, 1]$.

A Markov chain is said to be irreducible if it is possible to get to any state from any state. This is exactly the same thing as saying that the underlying graph is strongly connected; M is indeed irreducible as in our definition above.

We say that a state i has period h if any return to state i must occur in multiples of h time steps. This coincides with the notion of periodicity in graphs and with the number of eigenvalues in Proposition 1.17. Thus, a Markov chain is aperiodic if and only if M is primitive.

The distribution of probability at time $k+1$ follows the recursion

$$x_{k+1} = Mx_k$$

with x_0 the initial distribution.

Since M is Markov transition, it is stochastic: the sum of the columns is equal to 1. This means that 1 is an eigenvalue of maximum modulus. The equation above is then equivalent to the Power method and thus $x_k \longrightarrow \pi$ where $\pi = M\pi$ is the stationary distribution of the Markov chain.

Example 1.21 (Guinea Pig).

1.3.2 Gossipping

Every node is given a real, this gives a vector x_0, we want to exchange with our neighbors to compute the mean over the network.

Let M be a matrix with the same support (non-zero elements as A). We consider the iterations

$$x_{k+1} = Mx_k.$$

What can we do to converge to the average?

If M is doubly-stochastic (row and column sum equal to 1), $\rho(M) = 1$, $v = (1, .., 1)$, $w = (1, .., 1)/n$. Thus

$$x_k \longrightarrow x_{ave}$$

where x_{ave} is the age of the values of x_0.

1.4 The Graph Laplacian

1.4.1 Introduction

The adjacency matrix captures well the information of the graph and enables to prove a lot of results. But complementary results can be derive by considering *operators* on the graph such as the *Laplacian*.

Suppose the ϕ_i is the amount of heat at node i; according to Newton's law of cooling, the heat transferred from i to j is proportional to $\phi_i - \phi_j$. Thus, the heat evolution follows:

$$\begin{aligned}
\text{for all } i, \quad \frac{\mathrm{d}\phi_i}{\mathrm{d}t} &= -\kappa \sum_{j \in \mathcal{N}_i} (\phi_i - \phi_j) \\
&= -\kappa \sum_{j \in V} A_{ij} (\phi_i - \phi_j) \\
&= -\kappa \left(d_i \phi_i - \sum_{j \in V} A_{ij} \phi_j \right) \\
&= -\kappa \sum_{j \in V} \underbrace{(I_{ij} d_i - A_{ij} \phi_j)}_{:= L_{ij}} \phi_j
\end{aligned}$$

with I the identity matrix.

By introducing the diagonal *degree* matrix $D = \mathrm{diag}(d_1, .., d_n)$, we get

$$\begin{aligned}
\frac{\mathrm{d}\phi}{\mathrm{d}t} &= -\kappa (D - A) \phi \\
&= -\kappa L \Phi
\end{aligned}$$

and hence L control the evolution/mixing over the graph.

Definition 1.22 (Laplacian matrix)**.** For an *undirected unweighted*[4] graph $\mathcal{G} = (V, E)$, the Laplacian is the $n \times n$ matrix defined as

$$L_{ij} = \begin{cases} d_i & \text{if } i = j \\ -1 & \text{if } i \neq j \textit{ and } j \in \mathcal{N}_i \\ 0 & \text{otherwise.} \end{cases} .$$

[4] most properties developed after are also true for directed and weighted graphs but the definition has to be defined more carefully to maintain the semi-definite structure.

Definition 1.23 (Incidence matrix)**.** For an *undirected unweighted* graph $\mathcal{G} = (V, E)$, the incidence matrix is the $|E| \times n$ matrix defined as

$$E_{(i,j)v} = \begin{cases} 1 & \text{if } v = i \\ -1 & \text{if } v = j \\ 0 & \text{otherwise.} \end{cases} .$$

It is direct to show that $L = E^\top E$, thus L is *positive semi-definite*. The study of Laplacian matrices is thus based on the positive semi-definite matrix theory.

1.4.2 *Spectral Properties of Positive Semi-Definite matrices*

We recall that a *symmetric*[5] matrix is positive semi-definite if and only if

- $x^*Lx \geq 0$ for all $x \neq 0$;
- or (equivalently) if it can be written as M^*M; in which case M is called the squared root.

[5]The symmetry is sometimes eluded in textbooks, but it is often a source of problems.

Lemma 1.24. *All the eigenvalues, the trace, the determinant, of a positive semi-definite matrix are non-negative.*

Proof. As an exercise. □

An important tool to investigate eigenvalues is the formulation of the eigenvalue problem as an optimization program.

Theorem 1.25 (Rayleigh quotient)**.** *Let M be a symmetric matrix and let x be a non-zero vector that maximizes the* Rayleigh quotient *of M*

$$\mathcal{R}_M(x) := \frac{\langle Mx; x\rangle}{\langle x; x\rangle}.$$

Then, x is an eigenvector of M associated with eigenvalue $\mathcal{R}_M(x)$, which is the largest eigenvalue of M.

Proof. Let us consider without loss of generality the set of unit vectors; since it is a closed compact set, the maximum of $\mathcal{R}_M(x)$ is attained. Let us look at optimality conditions:

$$\begin{aligned} \nabla \frac{\langle Mx; x\rangle}{\langle x; x\rangle} &= \frac{2Mx\langle x; x\rangle - \langle Mx; x\rangle(2x)}{\|x\|^2} = 0 \\ \Leftrightarrow Mx &= \underbrace{\frac{\langle Mx; x\rangle}{\|x\|}}_{\in\mathbb{R}} x \end{aligned}$$

□

This can be further generalized by the following result.

Theorem 1.26 (Courant-Fisher theorem)**.** *Let M be a $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq .. \leq \lambda_n$ and associated eigenvectors $\psi_1, \psi_2, .., \psi_n$. Then,*

$$\begin{aligned} \lambda_k &= \min_{S\subseteq\mathbb{R}^n, \dim(S)=k} \max_{x\in S} \frac{\langle Mx; x\rangle}{\langle x; x\rangle} \\ \text{or equivalently,}\quad \lambda_k &= \min_{x\perp\psi_1,\psi_2,..,\psi_{k-1}} \frac{\langle Mx; x\rangle}{\langle x; x\rangle} \end{aligned}$$

for any $k = 1, .., n$.

1.4.3 *Spectral Properties of the Laplacian*

Now let us look at the spectral properties of the Laplacian of a graph.

Theorem 1.27. *Let $\mathcal{G} = (V, E)$ be an undirected unweighted graph and denote by $\lambda_1 \leq \lambda_2 \leq .. \leq \lambda_n$ the eigenvalues of the Laplacian L of $\mathcal{G}$. Then,* *a.* $\lambda_1 = 0$*;*

b. $\lambda_2 > 0$ *if and only if the graph is connected.*

Proof. *a.* Since L is positive semi-definite, $\lambda_1 \geq 0$. As $\sum_j L_{ij} = 0$ for all i, $(1, 1, .., 1)$ is an eigenvector for the eigenvalue 0. Hence, $\lambda_1 = 0$.
b. By Courant-Fisher theorem, we have $\lambda_2 = \min_{x\perp(1,1,..,1)} \frac{\langle Lx;x\rangle}{\langle x;x\rangle}$.

$$\begin{aligned}
[Lx]_i &= \sum_j L_{ij}x_j = d_i x_i - \sum_{j\in\mathcal{N}_i} x_j = \sum_{j\in\mathcal{N}_i} (x_i - x_j) \\
\langle Lx; x\rangle &= \sum_{i\in V} x_i [Lx]_i = \sum_{i\in V} x_i \sum_{j\in\mathcal{N}_i} (x_i - x_j) \\
&= \sum_{(i,j)\in E} x_i(x_i - x_j) + x_j(x_j - x_i) = \sum_{(i,j)\in E} (x_i - x_j)^2.
\end{aligned}$$

Take $\|x\| = 1$ without loss of generality. Suppose that there is an $x \perp (1, 1, .., 1)$ such that $\frac{\langle Lx;x\rangle}{\langle x;x\rangle} = 0$, then, $\sum_{(i,j)\in E}(x_i - x_j)^2$ and thus $x_i = x_j$ for all $(i, j) \in E$. If the graph is connected this obviously means that $x_i = x_j$ for all $i, j \in V$ and thus that $x \propto (1, 1, .., 1)$ which contradicts $x \perp (1, 1, .., 1)$. □

The second (smallest) eigenvalue of the Laplacian is often called the *Fiedler value* or *algebraic connectivity* of the graph. Intuitively, the higher λ_2, the more connected the graph.

Exercise: Prove that is $\mathcal{G}$ has k connected components, $\lambda_1 = \lambda_2 = .. = \lambda_k = 0$ and $\lambda_{k+1} > 0$.

1.4.4 *Special Graphs*

The Complete graph K_n

Lemma 1.28. *The eigenvalues of the Laplacian of the Complete graph* K_n *are*

$$\begin{aligned}
\lambda_1 &= 0 \\
\lambda_2 = .. = \lambda_n &= n.
\end{aligned}$$

Proof. 0 is associated with $(1, 1, .., 1)$ as before. If $x \perp (1, 1, .., 1)$, this means that $x_i = -\sum_{j\neq i} x_j$. Then,

$$[Lx]_i = (n-1)x_i - \underbrace{\sum_{j\neq i} x_j}_{=x_i} = nx_i$$

thus $Lx = nx$ for all $x \perp (1, 1, .., 1)$. □

The Star graph S_n

Lemma 1.29. *The eigenvalues of the Laplacian of the Star graph* S_n *are*

$$\begin{aligned}
\lambda_1 &= 0 \\
\lambda_2 = .. = \lambda_{n-1} &= 1 \\
\lambda_n &= n
\end{aligned}$$

Proof. **Exercise.** □

The Ring graph R_n

Lemma 1.30. *The eigenvalues of the Laplacian of the Ring graph R_n are*

$$\lambda_k = 2 - 2\cos\left(\frac{2\pi k}{n}\right)$$

for $k = 1, .., n$.

The Path graph P_n

Lemma 1.31. *The eigenvalues of the Laplacian of the Path graph P_n are*

$$\lambda_k = 2 - 2\cos\left(\frac{\pi k}{n}\right)$$

for $k = 1, .., n$.

Thus they have a smaller algebraic connectivity than the ring.

1.5 Important notions in Graph theory

1.5.1 Bottlenecks in Networks

Example 1.32.

A proportion $p \in (0, 0.5)$ of the nodes is (dangerously) loosely connected to the rest!

Definition 1.33 (Isoperimetry number)**.** Let $\mathcal{G} = (V, E)$ be an undirected unweighted graph and let

- S be a subset of vertices V;
- $\partial S = \{(i, j) \in E : i \in S, j \notin S\}$;.

The *isoperimetry number* (or *Cheeger constant*) of the graph is defined as

$$\theta_{\mathcal{G}} = \min_{S \subset V : |S| \leq n/2} \frac{|\partial S|}{|S|}.$$

The lower $\theta_{\mathcal{G}}$, the more failure possible failures in the graph.

Example 1.34. On the graphs seen before, we have:

- Complete: $\theta_{K_n} = \frac{n}{2} \longrightarrow +\infty$
- Star: $\theta_{S_n} = 1$
- Ring: $\theta_{R_n} = \frac{4}{n} \longrightarrow 0$
- Above: $\theta_{A_n} = \frac{1}{pn} \longrightarrow 0$

Theorem 1.35 (Cheeger)**.** *Let $\mathcal{G} = (V, E)$ be an undirected unweighted graph. Then for any $S \subset V$,*

$$|\partial S| \geq \lambda_2(L)|S|\left(1 - \frac{|S|}{|V|}\right)$$

$$\text{and thus,} \quad \theta_{\mathcal{G}} \geq \frac{\lambda_2(L)}{2}.$$

Proof. Define χ as $\chi_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{elsewhere} \end{cases}$ and $\overline{\chi} = \chi - \frac{|S|}{|V|}$.

Since $\sum_i \overline{\chi} = 0$, $\overline{\chi} \perp (1, .., 1)$. Then, we have:

$$\begin{aligned} \langle \overline{\chi}; L\overline{\chi} \rangle &= \langle \chi; L\chi \rangle \text{ (since } \overline{\chi} \perp (1, .., 1)) \\ &= |\partial S| \text{ (since for all } i \in S, [L\chi]_i = d_i - \sum_{i \in S \cap \mathcal{N}_i} 1\,) \end{aligned}$$

and

$$\langle \overline{\chi}; \overline{\chi} \rangle = |S|\left(1 - \frac{|S|}{|V|}\right).$$

Finally,

$$\lambda_2(L) \leq \frac{\langle \overline{\chi}; L\overline{\chi} \rangle}{\langle \overline{\chi}; \overline{\chi} \rangle} = \frac{|\partial S|}{|S|\left(1 - \frac{|S|}{|V|}\right)}$$

which directly gives the first part. To get the second part, observe that

$$\theta_{\mathcal{G}} = \min_{S \subset V : |S| \leq n/2} \frac{|\partial S|}{|S|} \geq \min_{S \subset V : |S| \leq n/2} \lambda_2(L)\left(1 - \frac{|S|}{|V|}\right),$$

then the right hand side is minimal whenever $|S| = |V|/2$ which gives the second part. □

Example 1.36. On the graphs seen before, we have:

- Complete: $\theta_{K_n} = \frac{n}{2}$, $\lambda_2(L) = n$, this is tight.
- Star: $\theta_{S_n} = 1$, $\lambda_2(L) = 1$, off by $1/2$.
- Ring: $\theta_{R_n} = \frac{4}{n}$, $\lambda_2(L) = 2 - 2\cos(2\pi/n) \equiv 4\pi^2/n^2$, off by $n\pi^2/2$.

1.5.2 Planar Graph

If one want to draw a graph or to design a routing without crossings, we need to ensure that the graph at hand is planar.

Definition 1.37 (Planar graphs)**.** A graph $\mathcal{G} = (V, Edg)$ is said to be *planar* it it can be drawn without the edges crossing.

Example 1.38 (3 utilities problem of Dudeney, 1917). Also known as gas, electricity, & water distribution problem.

It shows that the *bipartite* graph with $3 + 3$ nodes, denoted by $K_{3,3}$, is not planar.

Example 1.39 (Complete graphs). For $n = 2, 3$, it is obvious.

- K_4 is *planar*
- K_5 is *not planar.*

These two examples actually describe all non-planar graphs by Kuratowski's theorem(Kuratowski, 1930).

Theorem 1.40 (Kuratowski, 1930)**.** *A graph is planar if and only if it does not contains $K_{3,3}$ or K_5* as its minors.

Example 1.41. Peterson graph is transformable into the K_5 graph.

Once a graph is drawn in the plan (whenever, it is drawn with curves, it can always be drawn by straight lines), its edges delimit zones for which be have Euler's formula.

Proposition 1.42. *Let $\mathcal{G}$ be a planar graph with n vertices, e edges, and f faces, then*

$$n - e + f = 2.$$

Graphical sketch of proof.

□

1.5.3 Coloring

The goal is to assign one color to each vertex so that each edge links two different colors.

Definition 1.43 (Coloring)**.** Let $\mathcal{G} = (V, E)$ be an undirected unweighted graph. A k-*coloring* is a function $c : V \longrightarrow \{1, .., k\}$ such that $c(i) \neq c(j)$ for all $(i, j) \in E$. If such a function exists for $\mathcal{G}$, we say that it is k-colorable. The minimal integer for which $\mathcal{G}$ is colorable is called its *chromatic number* $\chi_{\mathcal{G}}$.

A graph can be

- 2-colorable, iff it is bipartite;
- 3-colorable, finding out is an NP-problem;
- 4-colorable, e.g. if it is planar (famous 4 colors theorem proven by Appel & Haken in 1977 by investigating 1478 critical cases)
- etc.