CHAPTER 2 Game Theory

GAME THEORY is a set of analytical tools to understand the phenomena observed when decision-makers interact.

The *players* pursue well-defined objectives (they are *rational*) and take into account what they know of the other players' behavior.

A *game* is the description of the players, their possible actions, and their interest. The modelling/formalization is very important.

A bit of history:

- Traces since 1713 by Waldegrave, for the analysis of a card game;
- Renewed interest in the 1920s with chess analysis;
- Von Neumann's "On the theory of Games of Strategy"(von Neumann, 1928) in 1928 kickstarted the field;
- Nobel prizes (economy mostly) in 1994 (inc. John Nash), 2005, 2007, 2012, and 2015 (Jean Tirole).

This part is mainly based on:

- The "course in game theory" by Obsborne and Rubinstein (Osborne and Rubinstein, 1994)

2.1 Description and Vocabulary

2.1.1 Normal form

There is a finite set of *players* $\mathrm{P} = \{1, .., N\}$.

Each player i has a *set of actions* S_i and a *payoff function* $\mathrm{g}_i : \mathrm{S}_1 \times .. \times \mathrm{S}_N \to \mathbb{R}$.

Definition 2.1. A game in *normal form* is a tuple $\Gamma = (N, \mathrm{S} = \{\mathrm{S}_i\}, \mathrm{g} = \{\mathrm{g}_i\})$.

2.1.2 Pure/Mixed Strategy

In a *pure strategy*, each player i chooses *one* action $\mathrm{s}_i \in \mathrm{S}_i$. Then, it receives the payoff $\mathrm{g}_i(\mathrm{s}_1, .., \mathrm{s}_N)$.

If instead each player chooses randomly an action in S_i, it is called a *mixed* strategy. Mixed strategies will be considered later.

2.1.3 *Different types of games*

We will illustrate several types of fundamental games that capture the diversity of normal games. Each time, we will exhibit a two players game ($N = 2$) as they can easily be represented graphically and are the most basic and insightful examples in game theory.

They are typically represented as a table:

		Player 2	
		s_2	s'_2
Player 1	s_1	$(g_1(s_1, s_2), g_2(s_1, s_2))$	$(g_1(s_1, s'_2), g_2(s_1, s'_2))$
	s'_1	$(g_1(s'_1, s_2), g_2(s'_1, s_2))$	$(g_1(s'_1, s'_2), g_2(s'_1, s'_2))$

Common interest A game where the players have the same payoff: $g_i = g_j$ for all $i, j \in P$.

Example 2.2 (Activity in Grenoble). Alice and Bob want to do something together, either trail T or ski S with no preference.

$$S_A = S_B = \{T, S\} \text{ and } g_A = g_B = \begin{cases} 1 & \text{if } s_A = s_B \\ 0 & \text{else} \end{cases}$$

Zero-sum games A game where the player are antagonist: $\sum_{i=1}^{N} g_i \equiv 0$

Example 2.3 (Matching pennies). Alice and Bob both have a penny; they secretly turn it to heads or tails. If the pennies match, Alice wins 1E and Bob loses 1E (Bob gives 1E to Alice). If they are different Alice gives 1E to Bob.

$$S_A = S_B = \{H, T\} \text{ and } g_A = -g_B = \begin{cases} 1 & \text{if } s_A = s_B \\ -1 & \text{else} \end{cases}$$

Battle of the sexes Mix between common interest and zero-sum.

Example 2.4 (Meetup). Alice and Bob want to meet tonight; Alice prefers to meet at a bar; Bob prefers to meet at home.

$$S_A = S_B = \{B, H\},\ g_A = \begin{cases} 3 & \text{if } s_A = s_B = B \\ 1 & \text{if } s_A = s_B = H \\ 0 & \text{else} \end{cases},\ g_B = \begin{cases} 1 & \text{if } s_A = s_B = B \\ 3 & \text{if } s_A = s_B = H \\ 0 & \text{else} \end{cases}$$

Prisonner's dilemma It is a classic game where Alice and Bob are arrested and individually given the possibility to stay silent or cooperate.

$$S_A = S_B = \{S, C\},$$

$$g_A = \begin{cases} -1 & \text{if } s_A = S \text{ and } s_B = S \\ -3 & \text{if } s_A = S \text{ and } s_B = C \\ 0 & \text{if } s_A = C \text{ and } s_B = S \\ -2 & \text{if } s_A = C \text{ and } s_B = C \end{cases}$$

$$g_B = \begin{cases} -1 & \text{if } s_A = S \text{ and } s_B = S \\ 0 & \text{if } s_A = S \text{ and } s_B = C \\ -3 & \text{if } s_A = C \text{ and } s_B = S \\ -2 & \text{if } s_A = C \text{ and } s_B = C \end{cases}$$

It is a fundamental game in economy, notably for the creation of rules enabling the denunciation of coalitions between companies.

Game of chicken A lot like the prisonner's dilemma but penalizing a lot mutual cooperation.

$\mathcal{S}_A = \mathcal{S}_B = \{S, C\}$,

$$g_A = \begin{cases} -1 & \text{if } s_A = S \text{ and } s_B = S \\ -3 & \text{if } s_A = S \text{ and } s_B = C \\ 0 & \text{if } s_A = C \text{ and } s_B = S \\ -20 & \text{if } s_A = C \text{ and } s_B = C \end{cases}$$

$$g_B = \begin{cases} -1 & \text{if } s_A = S \text{ and } s_B = S \\ 0 & \text{if } s_A = S \text{ and } s_B = C \\ -3 & \text{if } s_A = C \text{ and } s_B = S \\ -20 & \text{if } s_A = C \text{ and } s_B = C \end{cases}$$

It is the game modeling mutually assured destruction: cuban missile crisis, evolutionary biology, etc.

Cournot competition Antoine Cournot (1801–1871) analyzed the spring water duopoly:

- Two firms produce an equivalent product ($N = 2$);
- Each firm decides of a production level $s_i \in \mathbb{R}$ for a cost $c_i(s_i)$;
- The selling price result from the demand vs offer, it is common to both firms and depend on the total production $p(s_1 + s_2)$.

The profit/payoff for company 1 is $g_1(s_1, s_2) = s_1 p(s_1 + s_2) - c_1(s_1)$; the one for company 2 is $g_2(s_1, s_2) = s_2 p(s_1 + s_2) - c_2(s_2)$.

The question is which quantity to produce?

2.1.4 Target of Game Theory

Analyze these games and more precisely:

- Which strategies are best?
- Are there equilibriums?

2.2 Analysis for pure strategies

Notations:

$\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times .. \times \mathcal{S}_N$

$\mathcal{S}_{-i} = \bigtimes_{j \neq i} \mathcal{S}_j$

$g = (g_i)_i$

2.2.1 Dominating strategies

Definition 2.5. A strategy $s_i \in \mathcal{S}_i$ is *dominated* if there is $t_i \in \mathcal{S}_i$ such that

$$\forall s_{-i} \in \mathcal{S}_{-i}, \quad g_i(t_i; s_{-i}) \geq g_i(s_i; s_{-i}).$$

It is *strictly dominated* if the inequality is strict.

A rational player never plays a strictly dominated strategy.

Definition 2.6. A strategy $s_i \in \mathcal{S}_i$ is *dominating* if for all $t_i \in \mathcal{S}_i$

$$\forall s_{-i} \in \mathcal{S}_{-i}, \quad g_i(s_i; s_{-i}) \geq g_i(t_i; s_{-i}).$$

It is *strictly dominating* if the inequality is strict.

It is unique from definition. If it exists, it is the only rational action.

Example 2.7. What should player 1 play in the following game?

		Player 2	
		A	*B*
Player 1	*A*	$(0,-2)$	$(-10,-1)$
	B	$(-1,-10)$	$(-5,-5)$

- What will play Player 2?
- Deduce what should play Player 1.
- Is it the best payment both player could have had?

If there exists *dominated strategies*, they can be eliminated successively from the game.

2.2.2 Nash Equilibrium

Definition 2.8. A strategy profile $s = s_1 \times s_2 \times .. \times s_N \in S$ is a *Nash Equilibrium* (NE) if

$$\forall i, \forall t_i \in S_i, \quad g_i(s_i; s_{-i}) \geq g_i(t_i; s_{-i}).$$

It is a global equilibrium (contrary to the local ones seen before). No player has a singular interest to deviate from his action. It is thus a good way to conclude an agreement.

2.2.3 Back to the examples

Are there Nash equilibriums in the following games?

Common Interest

		Bob	
		T	S
Alice	T	$(1,1)$	$(0,0)$
	S	$(0,0)$	$(1,1)$

Zero Sum

		Bob	
		H	T
Alice	H	$(1,-1)$	$(-1,1)$
	T	$(-1,1)$	$(1,-1)$

Battle of the sexes

		Bob	
		B	H
Alice	B	$(3,2)$	$(0,0)$
	H	$(0,0)$	$(2,3)$

Prisonner's dilemma

		Bob	
		Silent	Cooperate
Alice	Silent	$(-1, -1)$	$(-3, 0)$
	Cooperate	$(0, -3)$	$(-2, -2)$

Game of Chicken

		Bob	
		Silent	Cooperate
Alice	Silent	$(-1, -1)$	$(-3, 0)$
	Cooperate	$(0, -3)$	$(-20, -20)$

2.2.4 Nash Equilibriums and dominating strategies

- There can be no, one, or several NEs.
- If there is a strictly dominating strategy matching each player, it is the unique NE.
- By eliminating successively strictly dominated strategies, NEs are preserved.
- A profile of dominating strategies is a NE.

2.2.5 Equilibrium Selection

a)

		Player 2	
		A	B
Player 1	A	$(9, 9)$	$(-15, 8)$
	B	$(8, -15)$	$(7, 7)$

(A,A) and (B,B) are two NEs. If the player are risk-averse, they may prefer (B,B) even though the payoff is smaller. Indeed, if the other player does not play the NE, the loss is smaller with (B,B).

b)

		Player 2	
		A	B
Player 1	A	$(2, 2)$	$(1, 1)$
	B	$(1, 1)$	$(1, 1)$

(A,A) and (B,B) are two NEs but B is dominated for each player while A is strictly dominating. So (A,A) seems better.

c)

		Player 2	
		A	B
Player 1	A	$(2, 2)$	$(1, 2)$
	B	$(2, 1)$	$(1, 1)$

All states are NEs!

2.2.6 Application: Vickrey auctions (1961)

They are sealed-bid, second price auctions. There are N players, and player i:

- estimates the price of the object at v_i
- its action set is $S_i = \mathbb{R}_+$ and corresponds to its bidding
- if he wins the auction (his bid is the greatest), he will make a profit based on the difference between his estimation and his bid, otherwise he will make 0 profit
- mathematically, its payoff if $g_i(s_i, s_{-i}) = v_i - \max_{j \neq i} s_j$ if $s_i > \max_{j \neq i} s_j$ and 0 else

Such auctions are used for instance in advertisement bidding (eg. Google Ads), for mobile bandwidth acquisition (eg. FCC), etc.

Exercise: Show that $(v_1, .., v_N)$ is a Nash Equilibrium.

2.3 Mixed strategies

For some games, NEs *with pure strategies* do not exist; for instance, in Rock-Paper-Scissors.

Example 2.9.

2.3.1 Mixed games

Mixed Strategies Let $\Gamma = (N, S = \{S_i\}, g = \{g_i\})$ be a game in normal form and let us suppose that *each* S_i *is a finite set.*

Definition 2.10. A *mixed strategy* σ_i for player i is a probability distribution on S_i.

$$\sigma_i = (\sigma_i(S_i[1]), .., \sigma_i(S_i[n_i])) \in \Delta(S_i)$$

where $\sigma_i(S_i[j]) = \mathbb{P}[i \text{ plays the } j\text{-th action in his set}]$ and $\Delta(S_i)$ is the simplex on S_i.

Interpretation:

- Random strategy (eg in Rock Paper Scissors)
- Model for a large number of players

We note $\Sigma = \bigtimes_i \Delta(S_i)$ and $\Sigma_{-i} = \bigtimes_{j \neq i} \Delta(S_j)$.

Mixed game

- Each player plays a mixed strategy $\sigma_i \in \Delta(S_i)$.
- The probability that the global strategy $s = (s_1, .., s_N)$ is played is $\prod_j \sigma_j(s_j)$.
- For a global strategy $\sigma \in \Sigma$, the *expected payoff* for player i is

$$g_i(\sigma) = \sum_{s \in S} \left[\prod_j \sigma_j(s_j) \right] g_i(s).$$

With these definitions, $\Gamma = (N, \Sigma = \{\sigma_i\}, g = \{g_i\})$ is a *mixed game*:

- The players simultaneously choose a pure strategy $s_i \sim \sigma_i$
- They get payoff $g_i(s)$
- Each player tries to maximize its expected payoff

2.3.2 *Nash Equilibriums for Mixed Games*

Definition
Definition 2.11. A mixed strategy profile $\sigma = \sigma_1 \times \sigma_2 \times .. \times \sigma_N \in \Sigma$ is a *Nash Equilibrium* (NE) if

$$\forall i, \forall \tau_i \in \Sigma_i = \Delta(S_i), \quad g_i(\sigma_i; \sigma_{-i}) \geq g_i(\tau_i; \sigma_{-i}).$$

Example 2.12 (Rock-Paper-Scissors). (1/3,1/3,1/3) is a NE.

Nash's Theorem (1950)
Theorem 2.13. *All finite*[6] *games have (mixed) Nash Equilibriums.*

[6] *with finite number of actions*

Proof. To follow □

2.3.3 *Dominated Mixed Strategies*

Definition 2.14. A mixed strategy $\sigma_i \in_i$ is *dominated* if there is $\tau_i \in \Sigma_i = \Delta(S_i)$ such that

$$\forall \sigma_{-i} \in \Sigma_{-i}, \quad g_i(\tau_i; \sigma_{-i}) \geq g_i(\sigma_i; \sigma_{-i}).$$

It is *strictly dominated* if the inequality is strict.

Example 2.15.

While we could remove strictly dominated mixed strategy, this does not lead to a reduction of the states of the game. However, we are still able to remove strictly dominated *pure* strategies.

Proposition 2.16. *Let (Γ^k) be the sequence of games produced by eliminating strictly dominated pure strategies in Γ. Then, for all k, $NE(\Gamma^k) = NE(\Gamma)$.*

Example 2.17.

2.3.4 *Looking for mixed equilibriums*

Definition 2.18. For player i, $\sigma_i \in \Sigma_i$ is a *best response* to $\sigma_{-i} \in \Sigma_{-i}$ if

$$\forall \tau_i \in \Sigma_i = \Delta(S_i), \quad g_i(\sigma_i; \sigma_{-i}) \geq g_i(\tau_i; \sigma_{-i}).$$

The set of all best responses for an adversarial strategy $\sigma_{-i} \in \Sigma_{-i}$ is denoted by $BR(\sigma_{-i})$

The following result is obvious from the definitions.

Proposition 2.19. *$\sigma \in \Sigma$ is a (mixed) Nash Equilibrium if and only if for all i, $\sigma_i \in BR(\sigma_{-i})$.*

There is a nice relation between pure and mixed strategies in terms of best response. To study it, let use denote the *support* of a mixed strategy as $\mathrm{supp}(\sigma_i) = \{s_i \in S_i : \sigma_i(s_i) > 0\}$, i.e. the pure strategies that have a positive probability to be played.

Proposition 2.20 (Weak Indifference)**.** *For player i, an adversarial strategy $\sigma_{-i} \in \Sigma_{-i}$, and $\sigma_i \in \mathrm{BR}(\sigma_{-i})$, then*

$$\forall s_i \in \mathrm{supp}(\sigma_i), \quad g_i(s_i;\sigma_{-i}) = g_i(\sigma_i;\sigma_{-i}).$$

This mean that all pure strategies in support have the same payoff, equal to the payoff of the mixed strategy.

Proof.

$$g_i(\sigma_i;\sigma_{-i}) = \sum_{s_i \in S_i} \sigma_i(s_i) g_i(s_i;\sigma_{-i}) = \sum_{s_i \in \mathrm{supp}(\sigma_i)} \sigma_i(s_i) g_i(s_i;\sigma_{-i})$$

Then:

1) $g_i(s_i;\sigma_{-i}) \le g_i(\sigma_i;\sigma_{-i})$ since $\sigma_i \in \mathrm{BR}(\sigma_{-i})$;
2) Suppose that there is $t_i \in \mathrm{supp}(\sigma_i)$ such that $g_i(t_i;\sigma_{-i}) < g_i(\sigma_i;\sigma_{-i})$. Then,

$$\begin{aligned} g_i(\sigma_i;\sigma_{-i}) &= \sum_{s_i \in S_i} \sigma_i(s_i) g_i(s_i;\sigma_{-i}) \\ &< \sum_{s_i \in \mathrm{supp}(\sigma_i)} \sigma_i(s_i) g_i(\sigma_i;\sigma_{-i}) \text{ (by our supposition)} \\ &= g_i(\sigma_i;\sigma_{-i}) \text{ (since } \sigma_i \text{ is a probability vector)} \end{aligned}$$

which is a absurd.

Hence, $g_i(s_i;\sigma_{-i}) \le g_i(\sigma_i;\sigma_{-i})$ for all $s_i \in \mathrm{supp}(\sigma_i)$. □

The notion of indifference can be strengthened as follows.

Proposition 2.21 (Strong Indifference)**.** *For player i and an adversarial strategy $\sigma_{-i} \in \Sigma_{-i}$,*

$$\sigma_i \in \mathrm{BR}(\sigma_{-i}) \iff \begin{cases} (1) & \forall s_i, t_i \in \mathrm{supp}(\sigma_i), \quad g_i(s_i;\sigma_{-i}) = g_i(t_i;\sigma_{-i}) \\ (2) & \forall s_i \notin \mathrm{supp}(\sigma_i), \quad g_i(s_i;\sigma_{-i}) \le g_i(\sigma_i;\sigma_{-i}) \end{cases}.$$

Proof. The forward way is direct from the previous proof. The other way comes from noticing that (1) + (2) imply that $g_i(s_i;\sigma_{-i}) \le g_i(\sigma_i;\sigma_{-i})$ for all $s_i \in S_i$ and thus σ_i is a best response to σ_{-i}. □

Using once again the link between best responses and Nash Equilibriums, we have the following result.

Corollary 2.22. *The strategy $\sigma \in \Sigma$ is a (mixed) Nash Equilibrium if and only if for each player i:*

$$\begin{cases} (1) & \forall s_i, t_i \in \mathrm{supp}(\sigma_i), \quad g_i(s_i;\sigma_{-i}) = g_i(t_i;\sigma_{-i}) \\ (2) & \forall s_i \notin \mathrm{supp}(\sigma_i), \quad g_i(s_i;\sigma_{-i}) \le g_i(\sigma_i;\sigma_{-i}) \end{cases}.$$

Thus, in order to find Nash Equilibriums:

- Remove strictly dominated pure strategies

- Try all possible supports
- Find probabilities leading to indifferent payoffs

Example 2.23.

Example 2.24.

Example 2.25 (Braess's paradox).

2.3.5 *A proof of Nash's theorem*

Exercise

2.4 Two player games

In this section, we focus on the important case when $N = 2$. Then the game writes in normal form $\Gamma = \{2; (\Sigma_1, \Sigma_2); (g_1, g_2)\}$.

2.4.1 *Max-Mix strategies*

Definition 2.26. Let $\omega \in \mathbb{R}$. We say that player i guarantees a payment ω if he has a mixed strategy that pays at least ω against any adversarial strategy:

$$\exists \sigma_i \in \Sigma_i : \forall \sigma_{-i} \in \Sigma_{-i}, \quad g_i(\sigma_i; \sigma_{-i}) \geq \omega$$

that is to say

$$\max_{\sigma_i \in \Sigma_i} \min_{\sigma_{-i} \in \Sigma_{-i}} g_i(\sigma_i; \sigma_{-i}) \geq \omega.$$

In fact, by linearity of the expectation of the payoff, we can consider only pure strategies.

Proposition 2.27. *The maximal payoff that player i can guarantee is*

$$v_i = \max_{\sigma_i \in \Sigma_i} \min_{\sigma_{-i} \in \Sigma_{-i}} g_i(\sigma_i; \sigma_{-i}) = \max_{\sigma_i \in \Sigma_i} \min_{s_{-i} \in S_{-i}} g_i(\sigma_i; \sigma_{-i})$$

Definition 2.28. A (mixed) strategy $\sigma_i \in \Sigma_i$ is *max-min* if $\min_{\sigma_{-i} \in \Sigma_{-i}} g_i(\sigma_i; \sigma_{-i}) = v_i$

A max-min policy is not necessarily a NE but it can be a sensible policy if player i is *risk-averse* or if the other player is not rational.

Example 2.29.

2.4.2 *Zero-sum games*

In zero sum two players games, $g_1 = -g_2$.

Theorem 2.30. *Let Γ be a zero sum two players game. A strategy (σ_1, σ_2) is a (mixed) Nash Equilibrium if and only if it is max-min. Furthermore,*

$$\begin{aligned} v_1 = g_1(\sigma_1, \sigma_2) &= \max_{\sigma_i \in \Sigma_i} \min_{\sigma_{-i} \in \Sigma_{-i}} g_i(\sigma_i; \sigma_{-i}) \\ &= \min_{\sigma_{-i} \in \Sigma_{-i}} \max_{\sigma_i \in \Sigma_i} g_i(\sigma_i; \sigma_{-i}) \\ &= -v_2. \end{aligned}$$

The payment of a Nash Equilibrium is thus $(v_1, -v_1)$; v_1 *is then called the* value *of the game.*

2.4.3 Link with linear programming

Max-Min means taking one player side and optimizing against the other. Let A be a matrix such that $A_{i,j} = g_1(S_1[i], S_2[j])$.

Then, if player 1 plays x and 2 plays y, the gain for player 1 is $x^\top Ay$. The max min solution of this game is then

$$x^\star = \operatorname{argmax}_{x\in\Delta} \min_{y\in\Delta} x^\top Ay$$

Example 2.31.