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Master 1 AM
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Numerical Optimization

Final Exam

2021/2022
Duration: 2h

The exercises are independent. The questions all have the same weight in the grade. The third exercise gives bonus points.

EXERCISE 1: (10 Pts)

Let us consider the $\mathbb{R} \to \mathbb{R}$ function

$$f : x \mapsto \begin{cases} x^3 & \text{if } x \geq 0 \\ +\infty & \text{otherwise} \end{cases} .$$

1. What is the domain of f? Is $\mathrm{dom} f$ closed, convex?
2. Draw the epigraph of f. Is it closed? What does this mean for f?
3. Is the function f convex? strongly convex?
4. Where is the function differentiable? Is the gradient of f Lipschitz continuous on this interval?

Let us try to minimize f by gradient descent.

6. Take a constant $c \in [0, 1]$ and $y > 0$. Give the conditions on the stepsize γ to have

$$f(y - \gamma \nabla f(y)) \leq f(y) - cf(y).$$

7. Is is possible to guarantee a functional decrease at each iteration with a constant stepsize gradient descent method?
8. What stepsize policy would you implement to have a guaranteed convergence in terms of functional value?
9. Which method (seen in the course) for nonsmooth optimization could be employed here?
10. Show that the problem can be simplified with the change of variable $t \leftarrow x^{3/2}$.

EXERCISE 2: (10 Pts)

1. Consider the following optimization problem

$$\mathcal{P} : \begin{cases} \max_{x \in \mathbb{R}^n} \frac{1}{3} \sum_{i=1}^n x_i^3 \\ \text{s.t.: } \sum_{i=1}^n x_i = 0\,; \\ \qquad \sum_{i=1}^n x_1^2 = n\,. \end{cases}$$

   (a) Using Lagrange multipliers λ and μ, respectively, find all KKT points of problem $\mathcal{P}$ for arbitrary $n > 2$. Express all KKT points in terms of the multipliers λ^* and μ^*.
   Hint: Recall that $|x| = x\,\mathrm{sgn}(x)$, where $\mathrm{sgn} : \mathbb{R} \setminus \{0\} \to \{-1, 1\}$ is the sign function such that $x \mapsto -1$ if $x < 0$ and $x \mapsto 1$ if $x > 0$. Also note that such sign function is undefined at 0.

   (b) By examining second order conditions characterize the set of all maximizers for arbitrary $n > 2$, and find the largest value of the objective function that satisfies all the constraints.

2. Let $a \in \mathbb{R}^n$ and $L \subseteq \mathbb{R}^n$ a subspace of $\mathbb{R}^n$.

(a) One way to formulate the projection problem is

$$\mathcal{P}_1 \begin{cases} \min_{x\in\mathbb{R}^n} \frac{1}{2}\|x-a\|^2 \\ \text{s.t. } Ax = 0 \end{cases}$$

where $L = \{x \in \mathbb{R}^n | Ax = 0\}$.
Formulate the dual problem $\mathcal{D}_1$ and provide a geometric interpretation of the dual problem. Recall that $\text{Im}(A^\top) = \text{Null}(A)^\perp$.

(b) Another way to formulate the projection problem is

$$\mathcal{P}_2 \begin{cases} \min_{x\in\mathbb{R}^n} \|x-a\| \\ \text{s.t.:} x \in L \end{cases}$$

i. Write $\mathcal{P}_2$ as a minimax problem.

ii. Show that the dual problem associated to $\mathcal{P}_2$ can be written as,

$$\mathcal{D}_2 : \begin{cases} \max_y \langle a, y\rangle \\ \text{s.t.: } \|y\| \leq 1; \\ \qquad y \in M. \end{cases}$$

and specify the subset $M \subseteq \mathbb{R}^n$.
Hint: for any $u \in \mathbb{R}^n$,

$$\|u\| = \max_{\|y\|\leq 1} \langle u, y\rangle$$

EXERCISE 3: (Bonus)

Let c denote a non null vector of $\mathbb{R}^n$, x_0 of point of $\mathbb{R}^n$ and H a symmetric positive definite matrix of size $n \times n$. Show that the solution of the problem

$$\begin{aligned} \underset{x\in\mathbb{R}^2}{\text{minimize}} \quad & c^\top x \\ \text{subject to} \quad & (x-x_0)^\top H(x-x_0) \leq 1 \end{aligned}$$

writes

$$\bar{x} = x_0 - \frac{H^{-1}c}{\sqrt{c^\top H^{-1} c}}.$$